Su, J. H., & Ying, J. M. (2014).
Enhancing in-service mathematics teachers' professional knowledge with an HPM approach as observed in teachers' reflections. *Taiwan Journal of Mathematics Education*, *1*(1), 79-97.
doi: 10.6278/tjme.20140307.003

Enhancing In-service Mathematics Teachers' Professional Knowledge with an HPM Approach as Observed in Teachers' Reflections

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In this paper we discuss an HPM approach that may enhance in-service teachers' mathematics content knowledge and PCK at the same time. The method we used was to let four high school mathematics teachers read historical texts in mathematics, and ask them to write reports on their reflections. The historical texts consisted of four proofs of Heron's Formula provided by various scholars in history. The reports were then collected and analysed. We used the PCK model of Veal and MaKinster to explain the enhancement of the teachers' PCK. Their reflections indicated that their content knowledge, knowledge of students, and knowledge of instructional strategies improved. In addition, the vertical links from content knowledge to several attributes of PCK, as well as the horizontal links among those attributes were strengthened.

Keywords: Heron's Formula, history and pedagogy of mathematics (HPM), pedagogical content knowledge (PCK).

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Received：31 August 2013;
Accepted：7 March 2014.

蘇俊鴻、英家銘（2014）。
由教師反思中所見之在職數學教師專業知識增強的 HPM 進路。
臺灣數學教育期刊，1（1），79-97。
doi: 10.6278/tjme.20140307.003

由教師反思中所見之在職數學教師專業知識增強的 HPM 進路

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本研究討論一個由在職教師為主所組成的數學史討論班課程中之教學活動，這個課程採用數學史與數學教育（HPM）的進路，希望能同時提升在職數學教師的數學知識與學科教學知識（PCK）。在這個教學活動中，四位在職高中數學教師閱讀古代數學文本，理解後撰寫讀書心得與反思。古代數學文本的素材包含四位古代數學家對海龍公式證明的古文原文或現代英文翻譯。研究者使用海龍公式證明作為閱讀素材的理由有三點。第一，海龍公式通常被用來計算三角形面積，而現代教材運用餘弦定律的證明方法，展現符號代數的威力，但古代證明通常使用歐式幾何的證明策略，所以，兩種證明策略之間的張力，隱含了幾何與代數雙重表徵的連結問題，使古代證明的閱讀與反思成為我們用以檢驗學習者是否掌握，以及教師能否反思表徵形式之相關能力的極佳範例。第二，不同證明策略，會導致不同的理解困難，而教師本身在遇到這些困難之後，是否能意會到學生學習也會遇到類似的困難，也是可檢驗教師教學內容知識的視點。最後，海龍公式的古代證明會使用到各類不同的先備知識，使得閱讀者有可能利用海龍公式進行數學知識的縱深統整（vertical integration），進而強化教師的教學內容知識。教師們的反思在撰寫完畢之後由研究者分析，可以看到透過數學文本的閱讀，教師們對於海龍公式數學內容知識的理解，補強現行教科書之不足，能夠延伸至面積課程教學的內容。此外，透過幾何表徵的映照，突顯代數方法的簡便性，使得教師對於整個教材的結構脈絡有全面性的觀照。研究者利用 Veal 與 MaKinster 的學科教學知識模型來解釋教師教學知識的增強，而文本的閱讀正是由此模型的下層的內容知識，往中層對學生的知識與上層教學策略知識連結，透過數學內容知識的增加，進而強化 PCK 的其他特質，教師整體的 PCK 就能提升。研究結果顯示，這四位在職高中數學教師，他們的內容知識、對學生的知識，以及教學策略知識都有增長。同時，從內容知識到上層教學策略中各個屬性間的縱向連結，特別是與脈絡、評量、教學法、課程與社會文化等五個屬性的縱向聯結，以及那些屬性之間的橫向連結皆有增強。

關鍵詞：海龍公式、數學史與數學教育、學科教學知識

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收稿：2013 年 8 月 31 日；
接受刊登：2014 年 3 月 7 日。

I. Introduction

Approximately three decades ago, some educators and historians of mathematics began to explore the possible uses and effects of history in mathematics education at the primary, secondary and tertiary levels. International collaborations have been formed to investigate related topics, and many educators have come to believe that history plays a valuable role in mathematics teaching and learning (Fauvel & van Maanen, 2000). Recently, additional empirical studies have been conducted and positive results have been obtained regarding applying history in mathematics teaching and learning, about topics such as arithmetic, algebra, geometry, and calculus for students of various levels (Katz & Tzanakis, 2011; Liu, 2009). It would seem that more and more efforts have been exerted in the investigation and application of history in mathematics education.

However, if applying history in mathematics education is in fact a worthwhile cause, more mathematics teachers, both pre-service and in-service, should be provided some training in the history of mathematics and its applications in teaching and learning. Educators in Norway, the United States, and the United Kingdom have been trying to do exactly this in their respective countries (Katz & Tzanakis, 2011). In Taiwan, courses on the history of mathematics are provided in several teacher training institutions, some of which also engage in research projects that try to provide qualitative or quantitative evidence on applying history in teaching or teacher training. The study described in this paper was conducted in this context, discussing an approach for enhancing in-service mathematics teachers' professional knowledge in a graduate-level course.

II. Literature Survey and Rationale

In 1987, Lee S. Shulman first proposed the concept of pedagogical content knowledge (PCK), as a model for describing how teachers acquire new understandings of the content that they teach, and how these new understandings influence their teaching. He believed that PCK includes "an understanding of how particular topics, problems, or issues are organised, presented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987). A trend of research in related fields was started, especially research that involved describing and classifying the professional knowledge of teachers. Veal and MaKinster (1999) reported that numerous researchers have listed attributes or components of PCK and observed similarities in these PCK taxonomies. The three most important and recurring characteristics in the taxonomies are knowledge of students, knowledge of content, and knowledge of instructional strategies. However, researchers have not illustrated the relationships among the attributes. Veal and MaKinster then presented a hierarchical taxonomy of PCK attributes: the base level is content knowledge, the middle level is knowledge of students, and the top level is knowledge of instructional strategies. The three levels are embedded into one another. The top level is further subdivided into eight interconnecting attributes – context, environment, nature of science, assessment, pedagogy, curriculum, socioculturalism, and classroom management. Veal and MaKinster stressed that the development of those attributes does not need to be linear. Among the three levels, a lower level has more significance than a higher level, implying that having an extensive base of content knowledge is a necessary (but not sufficient) condition for a teacher's well-developed PCK.

The research on Pedagogical Content Knowledge also extends to studies on reading and writing. Discussions in literature about reading, writing, and argumentation mostly focus on the influence of reading and writing on students' concept learning or how, through the process of reading and writing, teachers understand students' thinking and learning of concepts. In fact, these discussions have to assume that teachers have a sound grasp of necessary content knowledge and PCK. For instance, "when a text involves propositions and proofs, it conveys the necessary logical relations between concepts; that the same concept may have different statuses (hypothesis, conclusion, lemma, or corollary), that different proof strategies imply distinct mathematical thinking…, are all features we have to consider when analysing texts of proofs" (Yang, 2004). This may adequately explain that in guiding text reading and writing, a teacher's knowledge about the text has to include a well-integrated system of content knowledge and knowledge of other related representations, so that teaching can meet the need of teaching goals and that of students' learning. The consideration responds to the idea of the importance of enhancing teachers' content knowledge and PCK.

Fan (2003) believed that teachers' self-improvement of professional knowledge relies primarily on reflections after teaching and interactions among colleagues; the significance of textbooks is only to the next. But let us recall the term coined by Felix Klein: "double-forgetting" – first forgetting school mathematics on going to college and then having to forget higher mathematics on going back to school as a teacher (Freudenthal, 1973). Thus, teachers' basis for reflection, beside interactions with colleagues, is ironically only at-hand textbooks. However, different versions of textbooks in Taiwan do not have salient content distinctions due to market considerations. Sometimes the explanations are even identical. Therefore, reading textbooks published by different publishers can seldom help teachers reflect on their teaching. The help current textbooks and teachers' handbooks can bring to their self-improvement is in fact minimal.

The difficulty mentioned in the previous paragraph may be coped with through reading historical mathematics texts. The study of an historical text, which is closely related to exploring the genesis or evolution of a mathematical concept, usually reveals aspects that cannot be found in the modern and refined definitions of the concept. From a teacher's point of view, this broader understanding offers ideas for creating new didactic situations, or even helps the teacher foresee some of his or her students' difficulties and errors when they struggle to grasp the meaning of the concept's modern definition (Thomaidis, 2005). Therefore, following the trends in studying the relations between history and pedagogy of mathematics (HPM) and the applications of history in mathematics education, we shall point out that the growth of teachers' PCK can benefit from reading historical mathematics texts, that is, an HPM approach. Recent studies have suggested that courses in the history of mathematics in teacher training programmes might serve to improve the mathematical content knowledge as well as the PCK of pre-service primary school mathematics teachers and pre-service secondary school mathematics teachers (Clark, 2011; Smestad, 2011). For pre-service secondary school mathematics teachers, in particular, an HPM course could help, to a certain extent, pre-service teachers construct coherent curricula and motivate them to develop their own material (Clark, 2011; Tzanakis & Arcavi, 2000). From the literature mentioned in this paragraph, we can see that there have been some studies which show that reading historical sources may help teachers explore the evolution of mathematical concepts, foresee students' difficulties, or construct coherent lesson plans. Therefore, from a didactic point of view, the HPM approach benefits several aspects of a teacher's PCK.

However, most of the research on the application of history in mathematics teacher training addressed pre-service teachers. For in-service mathematics teachers, whom Fan and Freudenthal were concerned about in their writings, there seem to be very few studies related to enhancing their professional knowledge with an HPM approach. This rationale naturally led to our research question:

How does the study of the history of mathematics impact the mathematical and pedagogical content knowledge of in-service mathematics teachers?

In this paper, we shall illustrate a qualitative case study on a reading unit in a graduate-level course on the history of mathematics in which in-service mathematics teachers participated.

III. Methodology

1. Research setting

This case study was conducted in a graduate-level course on the history of mathematics taught in a recent year in the department of mathematics in one of the universities in Taiwan that has a long history of teacher training, and the participants in the course provided consent. The specific task (reading and reflecting on four proofs of Heron's Formula, see below) was given to the participants in a "seminar" that lasted four weeks, with three hours of reading, reflecting, and discussing in each week. Prior to the "seminar", historical texts were assigned to different members in the course to study. And then they presented the contents to the members in the seminar, and others would comment on those proofs and then write reflections. The written reports of their free reflections were then collected and analysed by the researchers according to the model of Veal and MaKinster, and the reflections were used as data in this study.

The sample in this study comprised four in-service high school mathematics teachers participating in this course. We shall refer to them with pseudonyms that are Romanisations of letters in the Greek alphabet: Alpha, Beta, Gamma and Delta. Their backgrounds and prior exposure to the history of mathematics are described as follows.

Teacher Alpha has a bachelor's degree in mathematics from a Normal University, and a master's degree in teaching (教學碩士) from the same university. He had taken courses about the history of mathematics at both the undergraduate and graduate levels. At the time of this study, he had already had nine years of combined teaching experience in a public junior school and a public senior high school in Yilan County.

Teacher Beta has a bachelor's degree in mathematics from a comprehensive university, and a master's degree in teaching (教學碩士) from a Normal University. He had taken a course about the history of mathematics only in the graduate level. At the time of this study, he had already had 10 years of combined teaching experience at a private senior high school, a public junior high school and a public senior high school in Taoyuan County.

Teacher Gamma has a bachelor's degree and a master's degree, both of which in mathematics from a Normal University. She had studied the history of mathematics at both the undergraduate and graduate levels. She used to work as a teaching assistant for three years in her alma mater before she became a school-teacher, and at the time of this study, she had already had twelve years of combined teaching experience as a teaching assistant in a university, and as a teacher at a public junior high school and, subsequently, at a public senior high school in Taipei City.

Teacher Delta has a bachelor’s degree in mathematics from a Normal University, and a master’s degree in teaching (教學碩士) from the same university. He had taken courses about the history of mathematics in both the undergraduate and graduate levels. At the time of this study, he had already had nine years of teaching experience, at both the junior and senior levels, in a comprehensive high school in Taipei County (currently New Taipei City).

As the reader can see, the four teachers hold degrees from different universities and have different backgrounds. The locations of the schools at which they have taught range from urban areas to rural towns in Taiwan. The lengths of their teaching experiences differ, but they are not novice teachers. They were all more or less exposed to the history of mathematics when they were studying in undergraduate or graduate levels.

Since both authors of this paper have experience in studying texts with these teachers, we decided to conduct “participant observation”, letting one of the authors, also a high school mathematics teacher, join the process of text-reading and reflection.

Participant observation is a methodology appropriate for studies of aspects of human existence; it is especially appropriate for exploratory studies and descriptive studies. In educational settings, this methodology has been used in observation of teaching and learning in levels ranging from pre-school to university (Jorgensen, 1989). In studies of mathematics education, it has been shown that the methodology is particularly helpful when exploring mathematical thinking (Carraher, Carraher, & Schliemann, 1985). Since this study explored mathematics teachers’ changes in their professional knowledge through their reflections, a participant observing in their in-class discussions would help identify meaningful reflections after the class. One of the limits of participant observation is that the participant rarely remains uninvolved with those being observed (Jorgensen, 1989). In this study, since the participating researcher also, in a few inevitable cases, contributed in the class discussions, the non-participating researcher could serve as an “objective” reader of the teachers’ reflections when they were analysed by both researchers. In other words, those teachers’ reflections were analysed by the participating author and the non-participating one to seek trustworthiness through triangulation.

In the following subsection, we introduce the central mathematical content that was studied by the teachers – four proofs of Heron’s Formula.

2. The task – Reading proofs of Heron’s Formula

The task given to the teachers was to read four different proofs for Heron’s Formula. The famous formula is typically expressed in its modern form:

$$\text{triangle area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, and c are three sides of the triangle in question, and s is half of the sum of the lengths of its three sides. In chronological order, the four proofs given to the teachers were the classical proof by Heron in his work *Metrica* (first century A.D.) in the form of its English translation, a proof by Mei Wending 梅文鼎 (1633~1721 A.D.) in his book 平三角舉要 (Elements of Planar Trigonometry, middle of the 18th century) in the original form in classical Chinese, a proof by Li Shanlan 李善蘭 (1811~1882 A.D.) in his book 天算或問 (Some Questions about Astronomy and Mathematics, 1867 A.D.) also in the original form in classical Chinese, and finally the proof in the text book *Elements of Plane and Spherical Trigonometry* (1859 A.D.) by Elias Loomis (1811~1889 A.D.) in modern American English.

But why Heron's Formula? Why did the lecturer of this course wanted to use this topic to improve in-service teachers' professional knowledge?

The first reason is about its dual nature. Heron's Formula is used to calculate the area of a geometrical figure, while modern textbooks use algebraic methods to demonstrate its validity. This dual nature suggests a possible link between geometric and algebraic representations of the same mathematical concepts. Therefore, it is an appropriate tool for surveying learners' and teachers' ability to reflect upon different forms of representations. In preparation of this course and the study, the lecturer and the two authors of this paper noticed that, after Heron's Formula was introduced to China by missionaries in the 1600s, several different proofs were developed in the following centuries, and the strategies of proving transformed from geometric to algebraic approaches, and eventually became what we see in high school textbooks today, which is a paradigmatic example for students to practise Laws of Sine and Cosine. In the same centuries, from the 1600s to the 1900s, representations of "trigonometry" were also transformed from the forms of Euclidean geometry to those of symbolic algebra. As can be seen, the different representations in those proofs are suitable for stimulating the teachers' reflections.

The second reason is an extension of the first one. For the in-service teachers and other students taking that graduate-level course in history of mathematics, reading historical proofs was an opportunity for them to have indirect "dialogues" with great mathematicians in the past, and in the process, they could form their own interpretations, free themselves from the limit of modern textbooks, compare the differences between geometric and algebraic strategies of proving, and understand the evolution and accumulation of mathematical knowledge. From a research point of view, this means that, with those different proofs in mind, the teachers might reflect upon students' possible difficulties or misconceptions when they encounter modern definitions of concepts and proofs of propositions about those concepts. In addition, the researchers would have an opportunity to check whether the teachers had had enough grasp of the mathematical knowledge and students' learning.

The third reason is related to the value of "vertical integration". The investigation of area and volume is always an issue for ancient mathematicians due to its relevance to the real world, so related topics are seldom excluded from a course about the history of mathematics. Triangle formulae, in particular, are related to mathematical contents studied at the elementary to senior high school levels. Heron's Formula, as well as its various proofs, may remind the teachers of how area formulae are demonstrated at the primary, secondary and tertiary levels. As can be seen from proofs listed below, they make use of concepts from elementary school mathematics (such as simple arithmetic and area formulae), junior high school mathematics (such as Euclidean propositions and arithmetic of proportions) and senior high school mathematics (such as algebraic manipulations and trigonometry). The teachers would be reminded, or taught, how lower-level mathematical concepts can be used to solve higher-level problems, and how they should connect those mathematical concepts from different levels for their students. That is what we mean by "vertical integration" in both the content and pedagogical knowledge of different levels of school mathematics. To sum up, it is the very transformation of proving strategies and representations, the possibilities to foresee students' difficulties from past proofs, and the value of vertical integration, that caused the lecturer and the researchers to use the proofs of Heron's Formula as a tool to investigate how reading ancient mathematical texts might improve teachers' content knowledge about the formula itself, and in turn stimulate the enhancement of their PCK.

In what follows we present outlines of the four proofs for Heron's formula. The English translation of Heron's original proof is quoted from Fauvel and Gray (1987), while the other three proofs presented below are the authors' own interpretation.

(a) Heron's proof

In his *Metrica*, Heron discussed the methods to find the area of a triangle with the lengths of three given sides. He gave two ways: "[n]ow it is possible to find the area of the triangle by drawing one perpendicular and calculating its magnitude, but let it be required to calculate the area without the perpendicular" (Fauvel & Gray, 1987), which means, Heron intended to find the area under such a limitation. His proof was geometrical. See Figure 1. Let $\overline{BC}=a$, $\overline{AC}=b$, and $\overline{AB}=c$. Then it can be shown that $\overline{AE}=\overline{AF}=s-a$, $\overline{BF}=\overline{BD}=s-b$, $\overline{CE}=\overline{CD}=s-c$, and $s=(s-a)+(s-b)+(s-c)$, where $s=\frac{(a+b+c)}{2}$. Heron already knew that the area of ΔABC was $\frac{(a+b+c)}{2}\times r=sr$, where *r* is the radius of the inscribed circle. So all he had to do was to replace *r*. His method was to extend $\overline{CH}=s-a$, so that $\overline{BH}=s$. Once he derived the proportional equality $\frac{s^2}{s(s-a)}=\frac{(s-b)(s-c)}{r^2}$, he obtained $s^2r^2=s(s-a)(s-b)(s-c)$, and the proof was complete. Since he wanted to show the equality geometrically, he needed similar triangles. He chose ΔAEI and ΔBCL, and he showed their similarity by demonstrating that quadrilateral *BICL* can be inscribed in a circle, so its opposite angles were supplements to each other. Using the similarity, he obtained the proportions $\overline{BC}:\overline{CL}=\overline{AE}:\overline{EI}=\overline{CH}:\overline{DI}$ ($\overline{AE}=\overline{CH}$, $\overline{EI}=\overline{DI}$). Then he used properties of proportions to replace the segments, and he could successfully prove that equality and, finally, the formula. In his proving process, Heron actually made use of many propositions from Euclid's *Elements* (such as the triangle area formula "*sr*" and properties of similar triangles), thus demonstrating the applicability of Euclidean geometry.

(b) Mei Wending's proof

Mei Wending's proof was not his original idea. According to our investigation, the proof was presented in the 測量全義 (Complete Explanation of Measurements) written by the Jesuit missionary Giacomo Rho (Mei, nd/1993; Rho, 1631/2000) in classical Chinese in the middle of the seventeenth century, but the proof contained a small flaw. Mei corrected it and included the proposition in his 平三角舉要. This proof begins with the same way as does Heron's proof. As shown in Figure 2, he first extended $\overline{CH}=s-a$, so that $\overline{BH}=s$. Unlike Heron, however, the proportional equality Mei wanted was $\frac{s-b}{s}=\frac{r^2}{(s-a)(s-c)}$, and the similar triangles to obtain that equality were ΔBID and ΔBGH. Once he showed the similarity between the two triangles, he obtained $\overline{BD}:\overline{BH}=\overline{ID}:\overline{GH}=\overline{ID}^2:(\overline{GH}\times\overline{ID})$. From $\Delta CGH\cong\Delta ICD$, he then replaced $\overline{GH}\times\overline{ID}$ with $\overline{CH}\times\overline{CD}$, and the proof was complete.

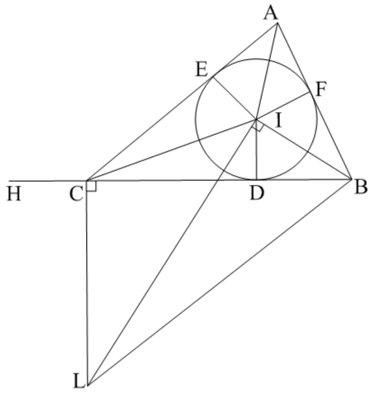

Figure 1 Diagram for Heron's proof of his formula for determining the area of a triangle.

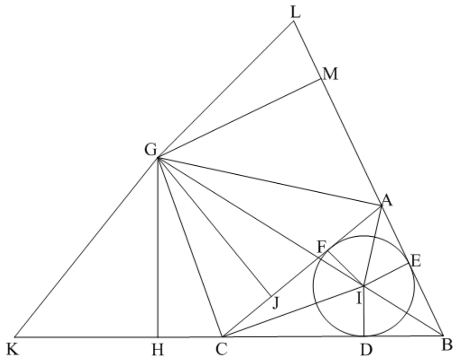

Figure 2 A reproduction of Mei Wending's diagram for his proof of Heron's Formula.

(c) Li Shanlan's proof

Li Shanlan's proof can be found in his 天算或問 (Li, 1867/2002). With three pairs of questions and answers, he justified $(s-a)(s-b)(s-c)=sr^2$. The formula followed from multiplying s and taking square roots on both sides.

As in Figure 3, let $\overline{AE}=\overline{AF}=s-a$, $\overline{BF}=\overline{BD}=s-b$, and $\overline{CE}=\overline{CD}=s-c$. Surprisingly, Li first constructed a height $\overline{AH}=h$, and showed the proportional equality $\overline{CD}:(\overline{AC}+\overline{CH})=\overline{BD}:(\overline{AB}+\overline{BH})=r:h$. Therefore, $\frac{\overline{CD}}{b+\overline{CH}}\times\frac{\overline{BD}}{c+\overline{BH}}=\frac{r^2}{h^2}$. Besides, $h^2=c^2-\overline{BH}^2=b^2-\overline{CH}^2$, so $(c-\overline{BH})(c+\overline{BH})=(b-\overline{CH})(b+\overline{CH})$. Applying the fact $\frac{c+\overline{BH}}{b-\overline{CH}}\times\frac{b+\overline{CH}}{c-\overline{BH}}=\frac{s}{s-a}$, he obtained $\frac{(s-b)(s-c)}{r^2}=\frac{\overline{BD}\cdot\overline{CD}}{r^2}=\frac{(b+\overline{CH})(c+\overline{BH})}{(b+\overline{CH})(b-\overline{CH})}=\frac{c+\overline{BH}}{b-\overline{CH}}=\frac{s}{s-a}$, and the proof was complete.

(d) The proof in Elias Loomis's *Elements of Plane and Spherical Trigonometry*

The *Elements of Plane and Spherical Trigonometry* (Loomis, 1859) was widely used in the United States in the nineteenth century, and was also translated into Chinese and used in schools founded by Christian missionaries in China (Li, 2005). See Figure 4. The proof involves applying a Euclidean proposition ($\overline{AC}^2=\overline{AB}^2+\overline{BC}^2-2\overline{BC}\times\overline{BH}$) from another text book by Loomis: the *Elements of Geometry, Conic Sections, and Plane Trigonometry*. Using this proposition, he obtained $\overline{BH}=\frac{c^2+a^2-b^2}{2a}$, and then $\overline{AH}^2=\overline{AB}^2-\overline{BH}^2=c^2-\frac{(c^2+a^2-b^2)^2}{4a^2}=\frac{4a^2c^2-(c^2+a^2-b^2)^2}{4a^2}$. The area of ΔABC is $\frac{\overline{BC}\times\overline{AH}}{2}=\frac{1}{4}\sqrt{4a^2c^2-(c^2+a^2-b^2)^2}$, which was then simplified using algebraic manipulations to $\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{(a+b+c)}{2}$.

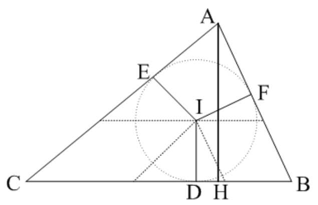

Figure 3 A reproduction of Li Shanlan's diagram in his 天算或問.

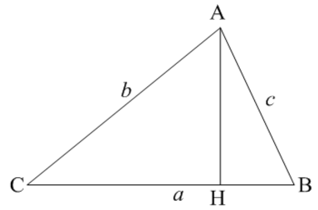

Figure 4 Diagram for Elias Loomis's proof of Heron's Formula.

IV. Data Analysis and Findings

With the collected reflections written by those teachers, we can now explain why reading ancient mathematical texts might enhance teachers' content knowledge and PCK, according to Veal and MaKinster's taxonomy. The reflections were originally written in Chinese. Quotations were translated into English by the authors from the teachers' reflections. As mentioned earlier, the teachers' reflections were analysed by the participating author and the non-participating one to seek trustworthiness through triangulation. Therefore, the excerpts of the reflections quoted in this section objectively represent to a certain extent the changes of the teachers. In what follows we list our findings about the teachers' enhancements and provide excerpts of the teachers' reflections as the evidence in our analysis.

1. Enhancements of mathematical knowledge and the connection between content knowledge and socioculturalism

From the previous section, we know that Heron's Formula was first proposed in an historical scene due to the need for solving real-world problems, and geometrical representations were used in early proofs. Those proofs expanded the teachers' understanding of the mathematical content knowledge on Heron's Formula. Teachers Alpha wrote,

> [W]hen I first saw Heron's original proof, on the one hand, I was alerted to the scant knowledge I had; on the other hand, I had a sudden realisation, because it was only natural that Heron's idea came from geometrical concepts and figures…[H]ardly any textbook version mentions Heron's original proof.

Veal and MaKinster’s terminology can be applied in describing the change in Teacher Alpha’s professional knowledge. First, the teacher’s mathematical knowledge was improved for he learned how to prove Heron’s formula with Euclidean geometry. Second, the connection between content knowledge and the knowledge of context and socioculturalism were enhanced. The reason we say this is that Teacher Alpha realised he had actually known that, in Hellenistic Alexandria, where Heron worked, there was no symbolic algebra (because it was invented in the eighteenth century), and all geometrical propositions were proved with Euclid’s approach and Aristotelian logic. Teacher Alpha linked the mathematical content and socioculturalism in history.

2. Enhancements of the awareness to different representations and structures of proofs

We also observed that all the teachers, except Teacher Beta, mentioned that they noticed the different representations – geometrical and algebraic – used in the texts. The same three teachers also reflected on how they might be able to use the characteristics and differences among the proofs in lessons, thus enriching their teaching repertoire. For instances, Teacher Gamma wrote:

> [Heron’s and Mei’s proofs] had the same basic strategy: first they used the inscribed circle to obtain the area of the triangle, and then they used proportions to obtain the relationships between rs and segments s, s-a, s-b, and s-c. Both proofs have only one inconvenience in teaching; namely, because we cannot see the role a ‘height’ plays, teachers cannot link ‘1/2 × base × height’ to Heron’s Formula. For this problem, Li Shanlan’s proof provides a solution.

This excerpt provides evidence that the teachers’ content knowledge on the structures of the proofs was enhanced.

3. Enhancements of the connection between the teachers’ content knowledge and assessment.

Assessment is another attribute that is connected to content knowledge. Teacher Delta stated that:

> [U]sing Pythagorean Theorem to prove Heron’s Formula may be able to prevent students from perceiving that Heron’s Formula is merely a procedural knowledge.

Teacher Gamma also wrote in another place about the aspect of assessment:

> Knowing the geometrical counterparts of *s*, *s-a*, *s-b* and *s-c* […] helps in problem-solving, such as in solving the [following] problem [that was used in an exam]: ‘Given $\triangle ABC$, $\overline{AB} = 4$, $\overline{BC} = 5$, and $\overline{CA} = 6$. Its inscribed circle touches three sides at points *D*, *E*, and *F*. If the area of $\triangle AEF$ and $\triangle BDF$ are *x* and *y*, respectively, then $x : y = ?$’

These reflections, in Veal and MaKinster’s terms, can be considered as improvements of the connection between content knowledge and assessment, because, after reading Heron’s proof, Teacher Gamma could see the help students might receive from those proofs, and she could identify the link between the proofs and assessments of students’ knowledge about a triangle and its inscribed circle.

4. Enhancements of the connection between content knowledge and pedagogy

Another enhancement, the connection between content knowledge and pedagogy, can be observed in Teacher Beta’s reflection on Li Shanlan’s proof:

> Li Shanlan seemed to narrate [his proof] from the end, using ‘reverse reasoning’ and ‘transitional objects’…This seems to be different from proving strategies in modern texts. In them, solutions of a problem often begin with the ‘givens’, and then through several ‘transitional objects’ [that are found or created], the problem is solved. But they never explain why those transitional objects are necessary. By contrast, Li Shanlan’s proof began with the ‘result’, and then investigates what ‘transitional objects’ were necessary to achieve this result; therefore, he created and proved them. This is why he arrived at this form of solution strategy, which is actually closer to students’ natural learning curve…[Li Shanlan’s proof] looks like Confucius’ *Analects*, in which the questions and answers between the master and the pupils are used to elaborate and solve problems.

As the reader can see, in this excerpt, Teacher Beta mentions the connection between content knowledge and pedagogy, since he compared the teaching strategies among modern textbooks, Li Shanlan’s proof and the *Analects*.

5. Enhancements of the awareness to students’ possible difficulties and its connection with content knowledge

The excerpt of the reflection in the previous subsection also addresses how students’ problems can be solved after the teacher recognises them. Teacher Delta, too, mentioned how he recognised students’ difficulties after reading those proofs:

> [T]he proofs of Heron's formula…are not so natural, in both the past and the present, because whatever method you use, they are all very tricky. Heron's original proof and Mei Wending's proof require using the radius of the inscribed circle, and you even have to construct very specific lines to fit the requirement…Although Li Shanlan used proportions, he still needed the inscribed circle. And for those who do not know proportional line segments that well, it is difficult to understand Li Shanlan's proof at first glance.

Thus, reading historical proofs also help teachers understand students' difficulties, because the teachers encountered the same difficulties. Therefore, we can see there is an enhancement of the knowledge of students, and its connection with content knowledge.

6. Enhancements of the connections between content knowledge and curricula

In addition, the implicit connection between geometrical and algebraic representations also helped the teachers consider the knowledge context and different strategies for arranging teaching material. Teacher Delta said,

> In the context of senior high school mathematics, it is very natural to verify Heron's Formula with Laws of Sine and Cosine, and we can conveniently check how well students learned the two Laws. If we leave this context, then it is not easy for students to deduce Heron's Formula naturally.

He also noticed the limitation of presenting the formula in this way:

> It seems to be a conclusion for Laws of Sine and Cosine, with no further room for development.

Clearly the connections between the knowledge of context and curricula were enhanced, since Teacher Delta could compare the historical proofs with those in modern textbooks, and realised not only why the author of textbooks present this formula in the context of trigonometry, but also that the textbook proof has no other development, suggesting that using historical proofs might introduced more relevant mathematical concepts in the curriculum.

7. Enhancements of the connections among content knowledge, pedagogy, and curricula

Further understanding of the content knowledge on Heron's Formula enabled the teachers to discover that it can be used as a bridge between the trigonometry contents in junior and senior high school levels, and Teacher Gamma even developed a feasible design for teaching materials. Teacher

Gamma stated that in Taiwan, the basic area formulae of triangles and rectangles taught in elementary school mathematics, the Euclidean propositions about triangles taught in junior high school mathematics, and trigonometry taught in senior high school, are taught separately with very few connections among them. Geometrical proofs of Heron's formula, if taught in the senior high school level, could serve as a bridge linking contents in the elementary and junior high school levels (such as 1/2 × base × height, triangle congruence properties, similar shapes, inscribed circles and so on) to senior high school mathematics. Those proofs in turn will help students understand the nature and value of the algebraic Laws of Sine and Cosine in a geometrical manner. Finally, through the learning process, all triangle area formulae could be summarised with trigonometry. This shows the teacher's professional growth through this HPM approach, as well as the connections among content knowledge, pedagogy and curricula. According to Veal and MaKinster's model, these connections are vertical connections among attributes, as are several other enhancements described in the previous subsections.

According to the analysis of teachers' reflections, the reading of historical texts of mathematics can improve teachers' comprehension of the mathematical content knowledge in Heron's Formula, complementing current textbooks. Those proofs deepened the teachers' understanding of triangle area and its formulae, so they could extend their teaching material for geometry. Furthermore, they felt the convenience of algebraic methods compared to geometrical representations, which shows that they became more perceptive to the panorama of the structural context of the mathematics curriculum.

V. Concluding Remarks

All the findings echoed three special effects of integrating history in mathematics, which are brought to our attention by Jahnke (2000): the first is replacement – it allows mathematics to be seen as an intellectual activity, rather than as just a corpus of knowledge or a set of techniques; the second is reorientation – it reminds us that mathematical concepts were invented and that this did not happen all by itself; the third is cultural understanding – it invites us to place the development of mathematics in the scientific and technological context in a particular time of certain societies.

Returning to Veal and MaKinster's PCK model, we believe that the two scholars proposed a valuable model but did not explain how those attributes of PCK influenced on or were connected to one another. Our study indicated that, by reading mathematical texts, the in-service teachers came to understand the genesis of Heron's Formula and the evolution of its proofs, realising the respective virtues and difficulties of geometrical and algebraic representations. In addition, the teachers' reflections revealed that the teachers became more knowledgeable of difficulties that might arise when teaching Heron's Formula and related concepts. As for the vertical connections between their mathematics content knowledge and the top-level attributes of their PCK, we can see that in several occasions in their reflections, they demonstrated that their content knowledge on Heron's Formula and its proofs connected with the attributes of context, assessment, pedagogy, curricula, and socioculturalism. Besides, we also see that the horizontal links among the five attributes were strengthened. In other words, an in-service teacher's mathematical content knowledge and PCK might both be enhanced with an HPM approach. This finding is consistent with the results of Smestad (2011) and Clark (2011) regarding pre-service teachers. We believe that this case study serves as a beginning of a direction in research exploring more methods for enhancing the professional knowledge of in-service teachers by using an HPM approach.

A final remark is about the correlation between a teacher's PCK and her maturity of the knowledge related to HPM. Although Veal and MaKinster mentioned that for the attributes of their PCK model, the usefulness, impact, and understanding would not be fully realised or integrated until a teacher has acquired several years of classroom experience (Veal & MaKinster, 1999), teaching experiences and seniority do not hold a salient position in the model. However, in this study we did observe that a teacher's PCK and her maturity of the knowledge related to HPM may be positively correlated. The four high school teachers have at least nine years of experience; thus, none of them were novices, and they easily understood the mathematical knowledge in the proofs. However, their reflections differed. Teacher Gamma, who had the longest experience in teaching and self-learning with an HPM approach, generated more ideas on how Heron's Formula is placed in the mathematical structure of the concept of the area and in the curricular structure in current high school textbooks than other teachers did. She also elaborated more on how it could be used in her own teaching, showing more relevance in PCK. By contrast, the other three teachers, who had less experience in self-learning and applying historical ideas in mathematics teaching, did not have as many opinions on how historical proofs could be utilised in teaching or curricular designs. It would seem that a teacher's PCK and his or her maturity in HPM (i.e., experience in learning and using historical texts and concepts in mathematics teaching and learning) are positively correlated. This study provides evidence of enhancements of PCK through an HPM approach, so a teacher with more experience in teaching and self-learning with historical material may have improved his or her PCK in the same time. Further studies can be conducted on this topic.

Acknowledgement

The authors would like to thank the two anonymous reviewers for their constructive comments. However, the authors take full responsibility for the opinions expressed and the accuracy of the contents in the paper.

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