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Developing Eleventh Graders' Conjecturing and Justifying Power on Generalization of Binary Number Patterns

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This study is due to investigate five eleventh graders of how they developing conjecturing and justifying power in the context of binary number pattern-finding problems. The framework of the research is situated in the social perspective which considers that the development of students' conjecturing and justifying power should be much relied on interactions, dialogues and argumentations within peers. Qualitative case study design and analysis strategy is adopted in the study for constructing a holistic viewpoint of students' mathematical conjecturing and justifying behavior. Research results show that students' inherent capabilities could be revealed when they are engaged in an atmosphere of conjecturing. Besides, students could not only apply specializing and generalizing strategies flexibly but also propose impressive justifications to make others convinced during the process of conjecturing. All of the findings seem to imply that relational understanding is crucial for students to develop their conjecturing and justifying power.

Keywords: conjecturing, generalizing, mathematical inquiry, specializing

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發展十一年級學生臆測及辯證能力—以二進位數字樣式探究活動為例

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本研究針對五位十一年級學生參與為期四週的課外數學探究活動進行觀察，藉以了解學生如何在解決數字樣式問題的過程中發展臆測及辯證能力。研究架構以社會學習面向為基礎，主要考量學生的臆測及辯證能力發展於同儕間之互動、對話與論證等學習行為。另本研究以個案研究為主進行資料蒐集及分析，從而瞭解學生數學臆測及辯證學習行為發展過程。研究結果顯示，學生在臆測情境氛圍中能夠有機會揭露並發展內在的數學能力，此外，學生在數學臆測的過程中不僅能彈性運用特殊化及一般化策略，更能提出令人驚豔的說明為其數學想法進行辯證。根據研究發現「關係式理解」對於學生臆測及辯證能力之發展至為關鍵。

關鍵詞：臆測、一般化、數學探究、特殊化

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I. Introduction

“Teaching to the test” is a globally common phenomenon of classroom teaching. In Taiwan, performance of paper and pencil tests of mathematics might still be the only way for assessing students’ achievements so far. Although Taiwanese students’ mathematical performance in TIMSS 2011 was remarkable, we still face the dilemma that high achievers were possessing non-positive attitudes towards mathematics learning (Martin, Mullis, Foy, & Stanco, 2012). Recently, the concept of “flipped classroom” is conceived as a promising thought of teaching approach. Such approach supports instructors playing their most important role of guiding their students to thinking deeply. Therefore, the student role could be shifted from a passive recipient to an active constructor of knowledge. This inspiration seems to echo the spirit of inquiry-based classroom. When students are engaged in inquiry activities, they could learn mathematics by means of “doing” autonomously, such as solving challenging problems, exploring relationships and patterns, forming conjectures and examining it, and communicating mathematical thoughts to others (Baroody & Coslick, 1993). Besides, teaching of mathematics should help students think actively and build new mathematical knowledge through problem solving. In particular, teacher’s role in choosing worthwhile problems and mathematical tasks is crucial (National Council of Teachers of Mathematics [NCTM], 2000). In addition, inquiry teaching could enhance mathematics understanding and thinking (Fennema, et al., 1996; Wood & Sellar, 1997). Furthermore, it could foster students’ creativity and ability of problem solving (Kwon, Park, & Park, 2006). Apart from that, there are four features for a tendency towards reform in mathematics teaching and learning in which are rich mathematical tasks, relating mathematics to real life experience and practices, learner-centered practice, and inquiry-based classrooms (Adler & Lerman, 2003). While inquiry based approaches have been drawing much attention in recent years globally, it is clear that conjecturing plays a crucial role in mathematical inquiry (Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007). Moreover, “communication can support students’ learning of new mathematical concepts as they act out a situation, draw use objects, give verbal accounts and explanations, use diagrams, write and use mathematical symbols” (NCTM, 2000, p. 61). Lin (2006) proposes that a good lesson must provide opportunities for learners to think and construct their own knowledge actively, and “conjecturing” might be the important strategy for involving learners to do so. It is not merely the core of mathematizing, but the driving force for mathematical proficiency. In short, we are inspired by synthesizing these remarkable insights and try to organize a study to see how students develop

conjecturing and justifying power actively in an inquiry-based classroom. Consequently, two research questions are proposed: (1) How do students apply conjecturing thinking for solving problems? (2) How do students develop conjecturing and justifying power when they are engaged in the number pattern generalizing activity?

II. Theoretical backgrounds

1. Mathematical inquiry

Recently, both science and mathematics learning standards focus on their attention of inquiry for either promoting students to construct knowledge actively in process of problem solving, reasoning, and communication or encouraging students to explore patterns and relationships in data analysis, formulating conjectures, logic thinking and solving non-routine problems. (American Association for the Advancement of Science [AAAS], 1990, 1993; NCTM, 1989, 2000; National Research Council [NRC], 1989). “Doing mathematics” should be considered as an inquiry process in mathematics learning. In order to develop mathematical thinking and the autonomy to solve challenging mathematical problems, students need to “do mathematics” (NCTM, 1989, 2000; NRC, 1989). Moreover, doing mathematics entails solving challenging problems, exploring patterns, formulating conjectures and examining them out, drawing conclusions and communication ideas, patterns, conjectures, conclusions and reasons (Baroody & Coslick, 1993). In brief, mathematical inquiry encourages students to construct mathematical knowledge actively and it must be underpinned by stances of constructivism. One of the hypotheses of constructivism is that knowledge is actively constructed by the cognizing subject, not passively received from the environment (Kilpatrick, 1987). Somehow, mathematical knowledge is not always constructed radically. Instead, sociocultural approaches contend that human thinking is inherently social in its origins (Kieran, Forman, & Sfard, 2001). Elbers (2003) replies the ideas with social perspective, he considers that when students are engaged in a community of inquiry, they could freely interact and collaborate with each other, and might have ample opportunities to make their own mathematical constructions and to discuss them in a social process of reflection. Empirical study result stands for these arguments as well. For example, Francisco (2013) holds a study with a group of six high school students working together on a challenging probability task as part of a larger, after-school, longitudinal study on students' development of mathematical ideas in problem-solving settings. The result shows that social settings, especially, collaborative activities can help promote students' mathematical understanding by providing opportunities for students to critically reexamine how they

make claims from facts and also enable them to build on one another's ideas to construct more sophisticated ways of reasoning.

2. Conjecturing and justifying

Conjecturing is an important part of an inquiry based approach (Cañadas, et al., 2007). Especially when individual confronts contexts of problems, he/she will actively propose conjectures, later, testing the conjectures, seeking counter examples for refuting it, and generalizing patterns of problems from systematical specializing strategy (Lakatos, 1976, 1978; Mason, 2002; Mason, Burton, & Stacey, 2010; Polya, 1954). Lakatos (1976, 1978) advocates that mathematics is quasi-empirical as he thinks that mathematics is a dialogue when people negotiate with it. In addition, mathematics is not flawless, it always needs to be renegotiated or reconstructed when facing possible challenges or much more stringent criteria. Lakatos concludes that theoretical knowledge can be established in the process of conjecturing and refutation. Mason and Johnston-Wilder (2004, p.141) argue that “mathematicians rarely solve the initial problems they set themselves. Most often they specialize, they conjecture, they modify and remodify until they find a problem they can do”. Above all, it is reasonable to acknowledge that conjecturing is an ongoing process which is built on specializing and generalizing as an ascent and descent (Polya, 1954). As a result, Mason et al. (2010) propose an idea to describe the conjecturing process. They consent that the process of conjecturing hinges on being able to recognize a pattern, or depending on being able to make a generalization. In short, the conjecturing process could be described as a cyclic process of articulating a conjecture, checking the conjecture, refuting/accepting the conjecture, and recognizing the pattern (see figure 1).

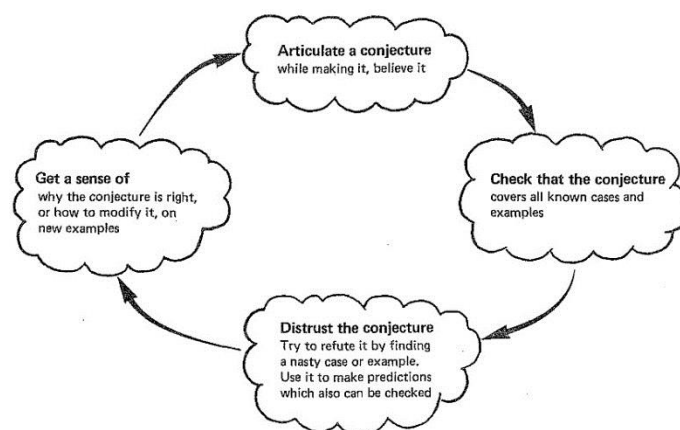


Figure 1 Conjecturing process. Reprinted from *Thinking mathematically* (p. 59), by J. Mason, L. Burton, & K. Stacey, 2010, Harlow, UK: Pearson Education Limited.

Moreover, generalizing and specializing are the two sides of a coin, in accordance with this view point, Mason (2002) points out two perceptions particularly, which are seeing the particular in the general and seeing the general through the particular. In addition, mathematics is perceived as the science of pattern and relationship. Consequently, exploring patterns, relations and functions is an essential focus of mathematics learning (AAAS, 1990; NTCM, 2000). Actually, generalizations are both objects for individual thinking and means for communication (Dörfler, 1991). For that reason, pattern-finding tasks in generalization can be considered as an important activity for getting students involved in a conjecturing atmosphere. Mason et al. (2010) synthesize these viewpoints, they think that specializing and generalizing are the backbone of the conjecturing process. Particularly, the problem solving phases, such as “entry”, “attack”, and “review” are owed much on specializing and generalizing (see figure 2). In addition, they see the “attack” phase is very much related to justifying and convincing, and it is also a crucial phase to seeing structural links.

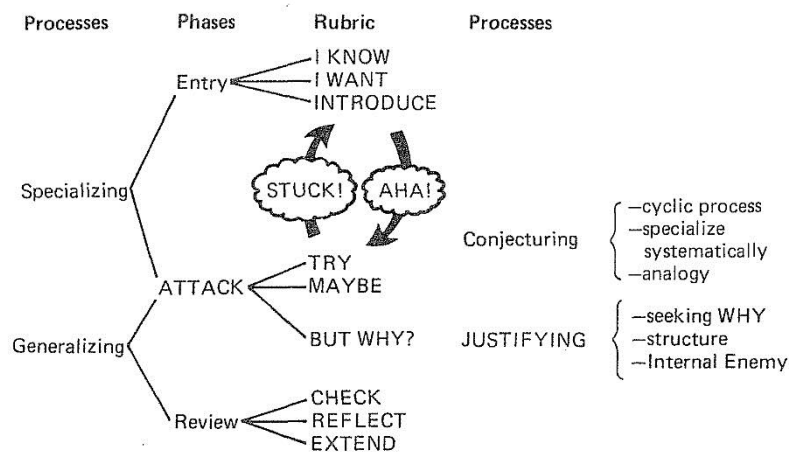


Figure 2 Backbone of conjecturing. Reprinted from *Thinking mathematically* (p. 77 & 95), by J. Mason, L. Burton, & K. Stacey, 2010, Harlow, UK: Pearson Education Limited.

Despite that the significance of conjecturing has been recognized by plenty of researchers (Davis, Hersh, & Marchisotto, 1995; Lakatos, 1976, 1978; Mason, et al., 2010), it could be recognized that evolving a conjecturing process in patterning approaches is one thing, justifying it for convincing others is quite another; even when students are able to generalize a pattern or a rule, few are able to explain why it occurs (Coe & Ruthven, 1994). Since that once you find the pattern, you need to state it carefully and clearly to convince yourself, convince a friend, and even to convince a skeptic (Mason et al., 2010). Mason (2002) further states that once a conjecture is made, it needs to be challenged, justified, and possibly reconstructed. Blanton and Kaput (2002) propose that justification induces a habit of mind

whereby conjectures in order to establish a generalization. Moreover, Stylianides (2007) concludes that justifying is using statements accepted by the classroom community and employs forms of reasoning that are valid and known to classroom community, as well as it is communicated with forms of expression that are appropriate and known to the classroom community. Consequently, conjecturing accommodates fruitful opportunities for reasoning in behalf of justifying conjectures. As a result, Zack and Graves (2001) adopt a sociocultural perspective to investigate discourse and its role in how children and teachers make meaning of mathematics in a fifth grade inquiry-based classroom for exploring the relationship between discourses and knowing. Therefore, “participating in a conjecturing atmosphere in which everyone is encouraged to construct extreme and paradigmatic examples, and to try to find counter-examples involves learners in thinking and constructing actively” (Mason & Johnson-Wilder, 2004, p.142). To sum up, when students are engaged in the context of inquiry-based conjecturing activity, they need to articulate the conjecture and specialize the conjecture systematically to validate it. Furthermore, they also have to propose an effective pattern based on the process of generalization to persuade themselves and others clearly and carefully. Even more, if the situation becomes complicated, they need to build on specializing and generalizing as an ascent and descent, in an ongoing process of conjecturing.

III. Methodology

1. Research setting

This study is conducted for developing eleventh students’ conjecturing and justifying power on pattern-finding problems. Since the study is implemented in an inquiry-based classroom, the pattern-finding activity is orchestrated in steps of presentation of a problem, whole-class discussion about the methods for solving the problem, and summing up by the teacher (Shimizu, 1999). Further, qualitative case study approach (Yin, 2009) is adopted in this study.

The pattern-finding activity is designed to support students’ development of conjecturing and justifying power. There are six tables of numbers (Figure 3) from 1~63 that are constructed by transforming of number systems from decimal to binary in students’ worksheet. For instance, decimal number 10 could be represented as the binary number 1010, that is, number 10 is equal to $(1)*2^3+(0)*2^2+(1)*2^1+(0)*2^0$. Among the six tables, table 1 contains the binary numbers with the first digit is 1. The rest may be deduced by analogy, such as table 5 contains the binary numbers with the fifth digit is 1. Hence number 10 can be found in tables 2 and 4 only. Some of the research subjects have

learned the binary system in the "information and technology" lessons, but not with this case. Hence, randomly placing the numbers in each table but not sequencing them in order might increase the degree of difficulty of generalizing the hidden pattern. In addition, there are three questions designed in this game: (1) What properties can you find from these six tables? (2) Can you induce any generality of the tables from the properties you found? (3) Can you propose any conjecture of how the game works and justify your conjecture?

No. 1

17	19	61	43	25	27	57	31
1	51	37	7	9	11	13	15
33	35	5	39	29	23	45	47
49	3	53	55	41	59	21	63

No. 2

50	51	10	43	58	15	62	63
2	27	6	7	54	22	14	38
18	19	11	23	26	3	30	31
34	35	59	39	42	55	46	47

No. 3

15	5	44	37	12	13	39	4
20	21	61	23	28	29	38	53
52	31	54	47	60	22	62	63
36	7	30	14	6	45	46	55

No. 4

46	13	26	63	28	11	30	31
12	62	42	43	44	45	29	47
56	8	58	59	60	61	41	27
57	9	10	24	40	25	14	15

No. 5

61	17	62	60	20	21	22	23
48	49	50	19	52	53	57	55
24	25	26	27	28	29	58	31
56	54	30	59	51	16	18	63

No. 6

58	33	59	32	36	51	41	39
40	38	42	43	56	48	34	47
63	49	50	37	52	57	54	55
44	53	61	46	60	35	62	45

Figure 3 Number tables for pattern-finding activity

2. Participants

Fourteen eleventh graders participate in a four-week extra-curricular program aimed to develop the power of conjecturing and justifying. Students are purposefully selected from seven different senior high schools with varied achievements. In addition, five of the students have chosen to be research subjects due to their fruitful performances. These schools are all located in the suburb of central Taiwan. The teacher of this study is one of the authors, who holds the viewpoint that students should construct mathematical knowledge actively and students' self-efficacy of math-learning might be fostered in the community of inquiry.

3. Number pattern generalizing activity

The activity is held throughout two phases, the first phase is started with a game asking a student to select a number from 1~63 and bear it in mind firstly, and then showing the student these six number tables sequentially for she/he to examine whether the selected number is in the table or not. In the end, the teacher notifies the student the accurate number she/he selected. After playing the game for several

rounds, students are engaged in personal construction for formulating ideas to initialize group discussion. At the end of personal session, we interview the students who have proposed conjectures in their own ideas; four of research subjects are purposefully select with maximum variation sampling strategy. Each critical case carries highly potential ideas of solving the hidden pattern of activity and can be seen a capable peer for leading collaborative learning. As regards the second phase, students are reorganized into four heterogeneous groups for following four selected representative students respectively (Louise, Isaac, Wendy and Bob) to find out the hidden patterns. Secondly, teacher walks around groups and poses productive questions for students. Finally, students' findings are integrated and validated through classroom community.

4. Data collection and analysis

Across a 4-week period, the whole activity procedures are videotaped and all the critical, extreme or unique, and revelatory cases are interviewed simultaneously while proceeding of group discussion at the second phase. Most of questions focus on the progress of students' development of specializing and generalizing power. All audio recordings are later transcribed verbatim. Further, field notes of class observation are taken during classroom video recordings and later expanded to help the researcher understand and orchestrate the story line of the study outcomes. Students' worksheets and reflections are also collected for interpretation of students' conjecturing behavior. Multiple evidence sources (Patton, 1987) are collected for constructing validity and triangulating evidence. Furthermore, all data are connected to a series of evidenced chain for enhancing reliability (Yin, 2009).

As regards data analysis strategy, first of all we generalize results systematically from the phenomenon observation (Strauss & Corbin, 1990). Secondly, students' mathematical conjecturing behavior are analyzed with theory driven. Competitive expositions are formulated from the results of critical cases. Finally, we take cross-case analysis and base on the level of findings of four groups respectively to construct a holistic view of students' conjecturing and justifying power on the pattern-finding activity.

IV. Data Analysis and Findings

Students are highly interested in the first phase of the activity. Most of them are amazed that how the teacher can get the right answer each time? Six students propose their primitive conjectures respectively. After that, four of them are selected to be a representative for leading the group discussion due to that they could propose their primitive conjectures clearly and carefully. Results show that their

conjectures might be clues of whether they could solve the puzzle or not. For instance, Louis finds the base numbers of six tables with different approaches, Wendy finds the algorithm of the binary number based on her findings of the pattern for the number tables, and Bod proposes a global perspective of articulating how the game works. The four representatives refine their primitive conjecture in the second phase through attacking and reviewing the problem. Especially, they also try to give their justifications to other group members from reflecting, extending and defending the conjecture, in which they go ascent and descent in an ongoing conjecturing process by the strategies of specializing and generalizing.

1. Results

Each group conducts their investigations in a specific way respectively. Despite they take different approaches to generalize patterns which are hidden in the activity, their results seem to be connected to each other although four groups are working independently.

Louise leads the group discussion based on her very first findings, and she is the first representative who presents their group discussion results. In the first phase, Louise finds that the smallest numbers in the six tables are 1, 2, 4, 8, 16 and 32 respectively, and any number between 1-63 might be composed of these six numbers. Isaac's group discovers the same findings as Louise's group does, but they go even further. They create a dichotomy method to sieve the guessed number out. Both Louise and Isaac find the numbers which are shown or not are followed by some particular patterns. Most of the students reorganize the sequence of numbers as ascent at once in the first phase of activity. Further, according to reconstruction of number tables, some groups apply the idea of checklist for generalizing the number patterns (see figure 7). Above all, using checklist for students is a facile approach which is also effective in observing hidden patterns. As in the case of this study, students could go a step further to see the relationship among the numbers.

3	5	7	9	11	13	15	
17	19	21	23	25	27	29	31
33	35	37	39	41	43	45	47
49	51	53	55	57	59	61	63

2	3	6	7	10	11	14	15
18	19	22	23	26	27	30	31
34	35	38	39	42	43	46	47
50	51	54	55	58	59	62	63

Figure 4 Part of Bob's group reconstruction of the sequence of numbers

As we compare the presentations of different groups, we could find that Wendy's group takes a quite different approach to generalize the pattern of numbers. They even find that numbers can be changed alternatively between binary and decimal systems. The followings are the transcriptions of interview with Wendy while she presents their work.

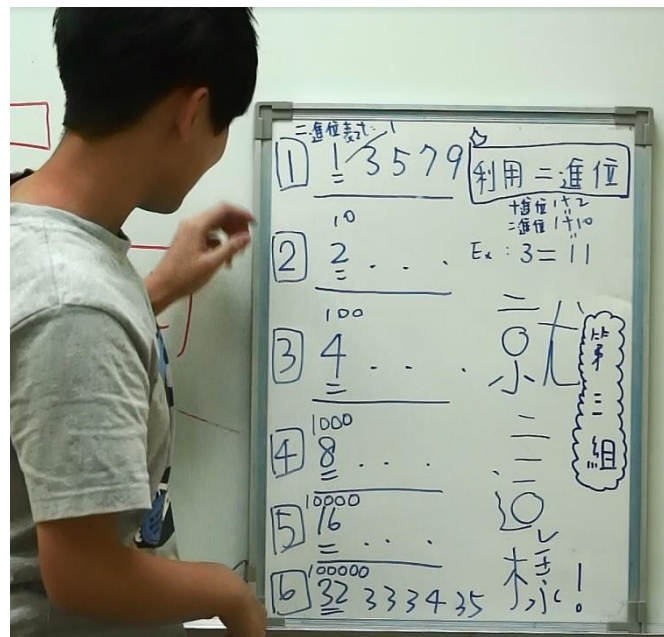


Figure 5 Presentation of Wendy's group

Teacher: Please tell us about your findings.

Wendy: Well, we reorganize numbers increasingly and find the first numbers in the six tables are 1, 2, 4, 8, 16, and 32 respectively. Then we transform these numbers from decimal to binary. Thus 2 can be represented as 10, 4 as 100, 8 as 1000, 16 as 10000, 32 as 100000.

.....

Teacher: Ok, what if we take 37 for example?

Wendy: 37 is 32 plus.....uh....., 4 and 1? Uh, thus 1 is in table 1, 4 is in table 2 and 32 is in table 6.

Teacher: And, how it relates to the binary system?

Wendy: Because binary system seems to be echoed to the pattern of numbers in the six tables.

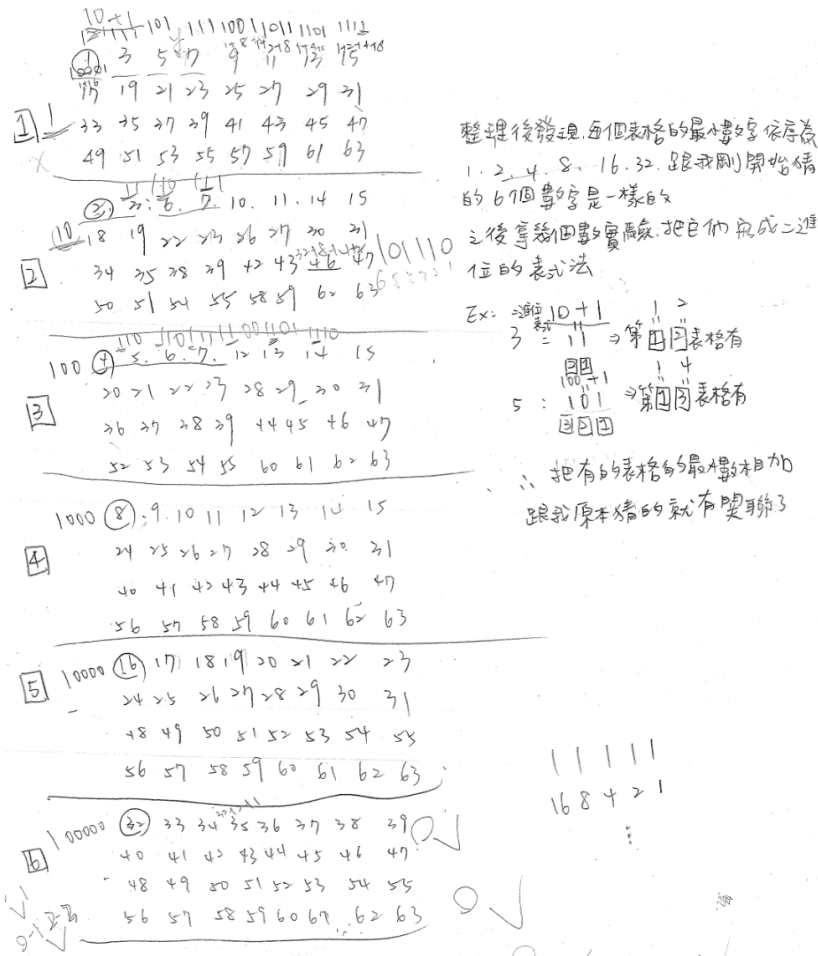


Figure 6 Wendy’s exploration of relation of number tables and binary system

We also notice that the idea of checklist is very much related with Wendy’s insight in the first phase of the study.

Teacher: Do you find the pattern of numbers in these six tables?

Wendy: May I write it down?

Teacher: Sure. You can talk while you are writing.

Wendy: I find a relation between the number and whether it is shown in these tables or not.

Teacher: Please show us how it works.

Wendy: For example, 37 is shown in tables 1, 3, 6. I see that table 1 to 6 represents numbers of 1, 2, 4, 8, 16, 32 respectively. Then I add 1, 4 and 34, and get 37.

Teacher: How do you find the rule?

Wendy: Well, I enumerate each possible number at first.

Teacher: And how do you find these six numbers 1, 2, 4, 8, 16, 32?

Wendy: It suddenly comes to my mind.

Teacher: Can you elaborate clearly?

Wendy: Uh, I think there are something must be added together while you are demonstrating.

Then I pick a number randomly for checking, and it is proved that I am right.

Teacher: Where do you get such ideas?

Wendy: By "intuition".

Teacher: Excellent.

Handwritten checklist table with columns for base numbers and checkmarks. The table is titled '第1阶段 Ex:' and '表格①'. It lists powers of 2 and their corresponding numbers, with checkmarks indicating which numbers are included in the sum.

Power of 2	Number	①	②	③	④
$2^1 = 1$	1	✓	✗	✓	✗
$2^2 = 2$	2	✗	✓	✓	✗
$2^3 = 4$	4	✗	✗	✗	✓
$2^4 = 8$	8	✗	✗	✗	✗
$2^5 = 16$	16	✗	✗	✗	✗
$2^6 = 32$	32	✗	✗	✗	✗

Figure 7 Wendy’s group expressing the generality by using checklist

Wendy’s conjectures could be seen as a response for Fischbein (1987), who recognizes that plausible conjectures are based on preliminary analysis. Hence conjectures might be seen as expressions of intuitions. Somehow we can see that Wendy does not take a wild conjecture, rather, she builds the final results at second phase according to her plausible conjectures. Besides, we find some evidences of how such plausible conjectures shed the lights for students’ pattern-generalizing. According to Wendy’s reflection after group discussion, there are some interesting outcomes as follows:

I start to list the number 1 to 9 at the beginning, and try to hypothesize the base numbers of these tables are 1, 2, 4, 6, 8, and 10, and then add them up if it’s checked in the table. Somehow I find that it is wrong. Then I try to hypothesize the base numbers are 1, 2, 4, 8, 16, and 32, respectively, and then I find it is consistent with my conjecture. I try to check some other numbers (small then 63), and find that it works. So I think I’ve found the secrets of the number pattern. In addition, I guess it must somehow relate to binary numbers thus I reconstruct the number tables and carry on for examining (see Figure 6). By the way, binary system is a topic which I learned from information technology class. It really surprises me that it can be applied to this game. [Wendy’s reflection at the second phase]

Apart from Wendy's group, Bob's group generalizes the pattern of the binary which hidden in the tables in a different perspective. They successfully use the checklist and make some marks on the list as signals of binary transformation. Further Bob's group members use the representation to explain how binary numbers and decimal numbers can be interchanged flexibly.

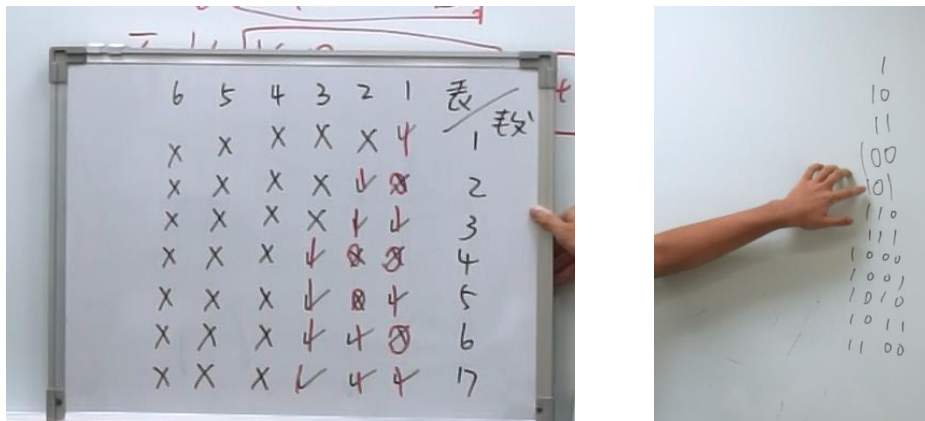


Figure 8 Bob generalizes a pattern of binary system

Bob: We use ticks for symbolizing the number which is checked in the number table and see it as binary digit 1. Similarly, we use crosses for standing the numbers which are unchecked in the number table and presume them as a binary digit 0. Then decimal numbers 2, 3, 4 and 5 can be interchange as 10, 11, 100, and 101.

Teacher: Bob, do you find any particular pattern of numbers related to numbers in the tables?

Bob: Yes, it is about the number whether is checked or unchecked in the table.

Teacher: Would you please show us?

Bob: Sure. We find that if you transform decimal numbers 1, 2, 3,...and 12 to binary numbers as 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011 and 1100. Then you can see all these numbers of the first digit with the pattern 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1 and 0 (e.g., the first digit of Table 1, and see it as vertically). This phenomenon explains why there are only odd numbers in the first table. And as you see these digits could correspond to decimals and to control whether it exists in the table or not.

Teacher: Excellent. How about the second digit of binary?

Bob: As you can see that the second digits of binary number start from decimal number 2. The pattern is of the form as 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1 and 0. This pattern corresponds to table 2 which makes decimal numbers echoing as x, 2, 3, x, x, 6, 7, x, x, 10, 11, x and so on. According to such phenomenon, we can make sense of the findings of Louise and Isaac.

Teacher: So do you find any relationship of transformation between decimal and binary?

June: Firstly, we see the six tables from the first to the sixth corresponding to decimal numbers of 2^0 , 2^1 , 2^2 , 2^3 , 2^4 and 2^5 . As the transformation between decimal and binary, for instance, if we transform binary number 10 to decimal we can add $1*2^1$ and $0*2^0$ then we get decimal 3. On the other hand, if we pick decimal 7, then we can see the number shows in tables 1, 2 and 3. Similarly, we can get decimal 7 equal to the result of $1*2^2+1*2^1+1*2^0$.

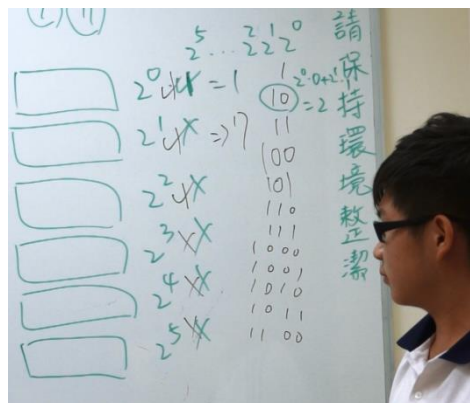


Figure 9 June (the second representative) illustrate the transformation within decimal and binary

There are two interesting twists of pattern-generalizing process of Bob's group. First one is that June asks for a reorganized number table by sequentially in the very beginning of second phase, but he has been asked for doing it by himself. It seems an ordinary action of student's working triggering a further exploration of their findings. After June reorganizes the number tables, he does a checklist with symbols of tick and cross. Meanwhile, when Bob sees the checklist, he shouts out suddenly "It's a binary number system". It seems that Bob's group finds the Holy Grail of this game; however, they are frustrated in solving the relation between binary and decimal.

June: I think it relates to addition. It can be found that number 1 is only checked with 2^0 , and 2^1 can be checked with decimal numbers 2 and 3. Further, 2^2 can be checked with decimal numbers 4, 5, 6 and 7 and so on (ref. Figure 9).

Bob: It looks like a binary system. (It seems not drawing attention.)

June: Numbers seem to be added by the powers of 2.

Bob: It looks like a binary system for me. (Bob tries to draw other's attention again.)

June: Why?

Bob: This is 000001($x \times x \times x \times \checkmark$), this is 000010($x \times x \times x \checkmark x$), this is 000100 ($x \times x \checkmark x x$), this is 000110 ($x \times x \checkmark \checkmark x$) and this is 000111($x \times x \checkmark \checkmark \checkmark$). (Based on the checklist.)

David: Oh! And then?

Bob: Yes, then 000111, 001000, 001001, 001010.

.....

Teacher: Now you find the rule for transform base-2 to base-10. Are the rules related to the tables?

Bob: Are these number tables related to the rule? Why?

Teacher: More specifically, do you see the hidden pattern of numbers within these six tables?

June: First of all, we need to rearrange numbers ascent in each table respectively. Then we can see an interesting phenomenon. For example, if a decimal number can be changed into a binary number 11, then it can be checked in table 1 and table 2. Again, if we get a binary 101, then it will be checked in table one and table three.

Teacher: Well done. Who finds this secret?

June: Bob does.

Teacher: Bob, how do you find it?

Bob: We can start the procedure from decimal number 1. Since 2 to the zero power is 1, then we can see that numbers in table 1 are formed in an discontinuous series bur patterned in 1, x, 3, x, 5, x, 7.....And the second table represents the 2¹ which is 2, the number series starts from 2, and with the pattern of 2, 3, x, x, 4, 5, x, x. Further, the third table represents the square of 2, that is 4. Thus the number series in table starts from 4, and with the pattern of 4, 5, 6, 7, x, x, x, x, 12, 13, 14, 15,

David: Bob, you mean binary is related to tables?

Bob: I can see the pattern but I still try to realize the reasons.

June: My question is why we need to multiply certain value with each digit of binary then add them up to get decimals? How does it work?

Bob: Base-2 and base-10, I think. I get the formula from IT class, but just don't know how to explain the tables and the interchange of binary number and decimal number.

表	2^0	2^1	2^2	2^3	2^4	2^5
0						
1	✓					
2	X	✓				
3	✓	✓	X			
4	X	X	✓			
5	✓	X	✓	X		
6	X	✓	✓	X	X	
7	✓	✓	✓	X	X	X
8	X	X	X	✓	X	X
9	✓	X	X	✓	X	X
10	X	✓	X	✓	X	X
11	✓	✓	X	✓	X	X
12	X	X	✓	✓	X	X
13	✓	X	✓	✓	X	X
14	X	✓	✓	✓	X	X
15	✓	✓	✓	✓	X	X

Figure 10 June’s work on generalizing the pattern of binary

In the end of discussion, Bob’s group does not produce any further exploration between tables and the algorithm of binary, which has been proposed at least four times in the discussion. The reason appears trivial as they just miss the key concept of place value for base-ten. NCTM (2000, p.78) proposes that “in prekindergarten through grade 2 all students should use multiple models to develop initial understandings of place value and the base-ten number system”. Even more, Ausubel (1968) considers that the most important single factor influencing learning is what the learner already knows. The knowing might be vicarious experiences in other academic subject areas or truly experiential knowledge developed from personal experiences outside school or in practical school subjects (Stillman, 2000). Although binary is the number system which students might not be familiar with, it frustrates Bob’s group for missing the ideas of place value, but students more or less tries to employ their knowing for inferring they do not know, so “to know is one thing, to do is another” is very much appreciated in such circumstances. Additionally, Wendy and Bob explore patterns of binary differently. Wendy takes the perspective from base-ten, while Bob base-two. Wendy applies the combination of base numbers for finding the pattern while Bob explores more profoundly with the combination of checklist, numbers in tables, and the powers of 2.

2. Discussion

(1) Specializing and generalizing are two sides of a coin

Specializing-generalizing might be one of the most important strategies while students are engaged in an inquiry-based conjecturing activity. According the viewpoint of Popper (1972), when individual is

frustrated in adversity with unknown situation; he or she will try to propose a tentative solution by following results of endless error trying. Even more, backward to previous process recursively when he or she is confronted with new problems. In the discussion of Bob's group, as June and David are lack of knowledge of binary, Bob provides them with specialized examples and try to generalize a pattern for transformation between decimal number and binary number according to the numbers' pattern in tables. Secondly, when they are confronted with the challenging problem, Bob and June offer many tentative solutions for giving explanations either in formulation of binary or numbers' appearing patterns. Just like Wendy, she tries to seek the base number of each table by several trying, and attempts to match the rule for checking the guessed number, and to generalize the pattern from the results of specializing, and apply the generality for addressing her findings by specializing the cases. As what is noted previously, it could be seen as the empirical evidences for responding to Mason (2002), that seeing the particular in the general and seeing the general through the particular, as well as a reply to Dörfler (1991), that generalizations are both objects for individual thinking and means for communication. But we conclude such stance premised much on the student who can freely employ with both strategies.

(2) Students' justifying power evolve within sociocultural perspectives

Wendy and Bob both propose the binary system for serving explanations of the pattern of the game which they find. Their finding progress is helped a lot with their episodic prior knowledge of binary. When their propositions are challenged, Wendy and Bob respond in different directions. Wendy can elaborate her findings not merely subtlety but also flexibly by using base numbers such as 1, 2, 4, 8, 16, and 32 for representing decimal number in the binary system. She takes an approach which starts from the perspective of decimal numbers to seek the binary transformation and exemplifies with a reasonable representation. But when Wendy is challenged to show the relation between binary and the game she just attributes to the intuition. On the contrary of Wendy, Bob takes the perspective of binary algorithm for explaining how decimal numbers can be arranged in the tables with certain patterns. Further, he exemplifies that how the arrangement connects to the interchange between binary and decimal numbers. When Bob is choked with the challenge to address the origins of the transformation he does not try giving any explanation immediately. Rather, he works with June back and forth iteratively at least four times in group's inquiring. As Isaac and Louise see their patterns clearly, they can apply strong statement for backing their persuasion without hesitation. In addition, Bob also tries do make sense of other groups' findings such as Isaac and Louise base on their conjecturing results. As the disclosure of Zack and Graves (2001), each child's ideas are made up in part of someone else's' ideas and in part of their own.

Additionally, their understanding can be connected within discourse. Such realization could also be verified in our research, students construct understandings not merely with discourse, rather, inquiry questions and justification are also employed as well. Furthermore, in Bob's case, he takes a different conjecturing subject in the second phase when he sees June's checklist, and he can see things in a unique way, such as he sees the digits of binary number in different perspective. Such phenomenon reminds us that Polya (1954, pp. 7-8) builds on specializing and generalizing as an ascent and descent, in an ongoing process of conjecturing:

....[an inductive attitude] requires a ready ascent from observations to generalizations, and a ready descent from the highest generalizations to the most concrete observations... First, we should be ready to revise any one of our beliefs. Second, we should change a belief when there is a compelling reason to change it.

Furthermore, Vygotsky (1978, p.86) proposes that the distance between the actual developmental level as determined by independent problem solving and level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. Apart from that, Schoenfeld (1989) argues that students' problem solving behavior not merely relies on their sense of mathematics as a discipline in the situation, rather it is driven by the beliefs which they possess. Undoubtedly, Bob is a capable peer, who is able to see things insightfully and flexibly to change his belief when there is a compelling reason to change it. Also June could be seen as a rival for Bob, such rival who is able to learn things quickly and always try to seek any opportunities for filling up the blank in an ongoing process of discourse, then making the results of discussion fruitfully. Somehow we could see students' mathematical conjecturing behaviors are much more relied on the socio-perspective, since "the goal of a sociocultural approach is to explicate the relationships between human action, on the one hand, and the cultural, institutional, and historical situations in which this action occurs, on the other" (Wertsch, del Río & Alvarez, 1995, p. 11).

V. Conclusions

1. Conjecturing process is based on specializing and generalizing

The results of this study disclose that, while students are engaged in an atmosphere of conjecturing, most of them could apply the strategies of specializing and generalizing flexibly for observing patterns. Based on the results of specializing and generalizing, some students might continue to make plausible conjectures and justify them confidently. The study is orchestrated in social settings, for that reason,

students could freely interact and collaborate with each other, and have ample opportunities to make their own mathematical constructions in conjecturing process. Therefore, collaboration within peers plays a vital role. Such collaboration could foster students to make sense of certain proposition which is bridged with dialogue and discourse.

2. Scaffolding is essential for justifying

Teacher's scaffolding is crucial for the development of conjecturing process, planning questions carefully could lead students to reveal and explore their conjecturing thoughts specifically. In brief, teaching is to create an environment for learning mathematics and orchestrate participants to engage in the context of inquiry. In addition, teaching is to elicit and interpret what students do and know (Cobb, 2000; Fennema et al., 1996). Besides, teachers need to conduct and organize students' findings and sequence the presentation order for helping students realize the phenomenon of problem from different perspectives. Due to most of the participants lacking experience of engaging in such inquiry-based conjecturing activity, some of them need to polish their addressing skill for helping others make sense of their justifications. Ernest (1995) notes that conversation or dialogue can be seen as a central metaphor for knowledge and mind, nevertheless, spoken words do not equal to thoughts in the mind. Although there is still a group missing the key concept for resolving the challenged issue, we notice that most of the students could infer analogy and prompts which are provided by the teacher. Furthermore, this activity seems to improve the research subjects' disposition towards learning mathematics.

3. Suggestions

We would suggest that this kind of activities should offer ample opportunities for students to enhance communication skills for further bridging discourse and understanding. Even more, it might be a good example for the integration between mathematics and information technology class, especially for the introduction of binary system. As students' former learning experience appears to be excessively focused on instrumental understanding without any extra energetic thinking, the study is orchestrated for developing pupils' conjecturing and justifying power as well as for facilitating students' learning of mathematics towards relational understanding (Skemp, 1987). Finally, the results of the study might offer some empirical evidence for the claim of Lin (2006), which is "Conjecturing is the centre and pivot of all phases of mathematics learning – including conceptualising, procedural operating, problem solving and proving, and provides the driving force for developing these phases of mathematics learning".

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